Solution to Assignment 2

Supplementary Problems

Note the notations. These problems are valid in all dimensions. Hence we do not use (x, y) to denote a generic point as we do in $\mathbb{R}^2$. Instead, here $\mathbf{x}$ or $\mathbf{p}$ are used to denote a generic point in $\mathbb{R}^n$.

1. Let S be a non-empty set in $\mathbb{R}^n$. Define its characteristic function χ_S to be $\chi_S(\mathbf{x}) = 1$ for $\mathbf{x} \in S$ and $\chi_S(\mathbf{x}) = 0$ otherwise. Prove the following identities:

   (a) $\chi_{A\cup B} \leq \chi_A + \chi_B$.

   (b) $\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B}$.

   (c) $\chi_{A\cap B} = \chi_A \chi_B$.

   Solution. (a) For $\mathbf{x} \in A \cup B, x$ must belong either to A or B. Hence $\chi_{A\cup B}(\mathbf{x}) = 1 \leq \chi_A(\mathbf{x}) + \chi_B(\mathbf{x})$. On the other hand, when $\mathbf{x}$ does not belong to $A \cup B$, $\chi_{A\cup B}(\mathbf{x}) = 0$ and the inequality clearly holds.

   (b) Exhaust all possible cases (1) $x \in A$ but not in B, (2) $x \in B$ but not in A (3) $x \in A\cap B$, and (4) x does not belong to A nor to B. In all these cases, the identity holds.

   (c) When $x \in A \cap B$, both $\chi_A(x)$ and $\chi_B(x)$ are equal to 1, hence their product is equal to 1. When x does not belong to A or B, one of $\chi_A(x)$ and $\chi_B(x)$ must be 0, hence the product becomes 0.